UNCC 2001 Comprehensive, Solutions

March 5, 2001

1. Compute the sum of the roots of $x^2 - 5x + 6 = 0$.

   (A) 3 **(B)** 7/2 **(C)** 4 **(D)** 9/2 **(E)** 5

   (E) The sum of the roots of the quadratic $ax^2 + bx + c = 0$ is $-b/a$ which, for this example, is 5. Alternatively, factor the quadratic into $(x-3)(x-2) = 0$ and find the roots.

2. Find the slope of the line connecting the two points that satisfy

   $$2x + 3y = 6 \text{ and } x^2 + y^2 = 36.$$

   (A) -1 **(B)** $-2/3$ **(C)** $-1/2$ **(D)** $-1/3$ **(E)** 0

   (B) The two points both belong to the line $2x + 3y = 6$ whose slope is $-2/3$.

3. Compute the sum of all the roots of $(2x+3)(x-4) + (2x+3)(x-6) = 0$.

   (A) 7/2 **(B)** 4 **(C)** 7 **(D)** 13 **(E)** none of **A**, **B**, **C** or **D**

   (A) Factor to get $(2x+3)(2x-10) = 0$, so the two roots are $-3/2$ and 5.

4. The radius of the circle given by

   $$x^2 - 6x + y^2 + 4y = 36$$

   is

   (A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 36

   (C) Complete the squares by adding 9 and 4 to both sides to get

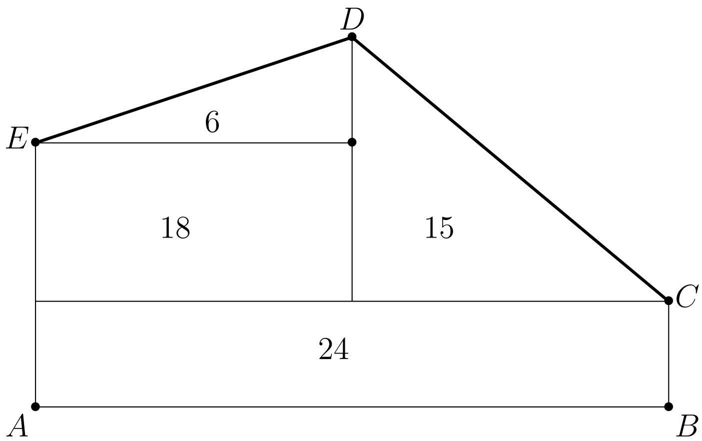
   $$x^2 - 6x + 9 + y^2 + 4y + 4 = 36 + 9 + 4 = 49 = 7^2.$$

   So the radius is 7.

5. What is the area of the pentagonal region $ABCDE$, where $A = (0, 0), B = (12, 0), C = (12, 2), D = (6, 7)$, and $E = (0, 5)$?

(A) 60 **(B)** 62 **(C)** 63 **(D)** 65 **(E)** 66

(C) Partition the region into two rectangles and two right triangles as shown, with areas $24, 18, 6$, and 15 respectively, for a total area of 63.



6. How many positive integers can be represented as a product of two distinct members of the set $\{1, 2, 3, 4, 5, 6\}$?

(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

(E) Count them one at a time. They are $2, 3, 4, 5, 6, 8, 10, 12, 15, 18, 20, 24$, and 30.

7. Two cards are selected randomly and simultaneously from a set of four cards numbered $2, 3, 4$, and 6. What is the probability that both cards selected are prime numbered cards? Express your answer as a fraction.

(A) 1/6 **(B)** 1/4 **(C)** 1/3 **(D)** 1/2 **(E)** 2/3

(A) Exactly two of the four cards have primes, so just one of the six pairs consists of primes.

8. How many positive integers less than one million have all digits equal and are divisible by 9?

(A) 5 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 18

(D) Suppose the number a has n digits each of which is d. Then a is divisible by 9 if and only if the sum $d + d + \cdots + d = nd$ of its digits is divisible by 9. Since $1 \leq n \leq 6$, one concludes that $d = 3, 6, 9$ are the only possibilities. With $d = 3$, we get the two numbers 333 and 333 333 divisible by 9. With $d = 6$, we get the two numbers 666 and 666 666 divisible by 9. With $d = 9$, we get the six numbers $9, 99, \ldots, 999\,999$ divisible by 9, so altogether there are $2 + 2 + 6 = 10$ solutions.

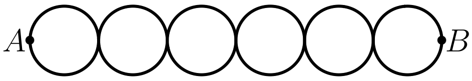
9. If $x^2+2x+n > 10$ for all real numbers x, then which of the following conditions must be true?

(A) $n > 11$ **(B)** $n < 11$ **(C)** $n = 10$ **(D)** $n = \infty$ **(E)** $n > -11$

(A) Complete the square to find that $f(x) = x^2+2x+n = (x+1)^2-1+n > 10$ if and only if $n > 11$. Alternatively, the minimum value of $x^2 + 2x + n$ occurs at the vertex of the parabola, whose x-coordinate is given by $-b/2a = -2/2 = -1$. Thus $f(-1) = (-1)^2 + 2(-1) + n > 10$ if and only if $n > 11$.

10. The diagram shows six congruent circles with collinear centers in the plane. Each circle touches its nearest neighbor(s) at exactly one point. How many paths of length 3π along the circular arcs are there from $A = (0, 0)$ to $B = (6, 0)$?

(A) 16 **(B)** 32 **(C)** 64 **(D)** 128 **(E)** 256



(C) The shortest path from A to B uses exactly six semi-circles. Selecting either the upper or the lower semicircle from each circle leads to $2^6 = 64$ paths from A to B.

11. What is the sum of the digits of the decimal representation of

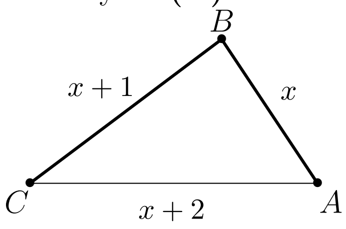
$$\frac{10^{27}+2}{3}?$$

(A) 80 **(B)** 82 **(C)** 84 **(D)** 86 **(E)** 87

(B) Note that $\frac{10^{27}+2}{3} = \frac{10^{27}-1}{3} + 1 = 333\ldots3 + 1 = 333\ldots34$, where the string has 27 digits. The sum of the digits is therefore $26 \cdot 3 + 4 = 82$.

12. Consider triangle ABC with $AB = x, BC = x + 1,$ and $CA = x + 2$. Which of the following statements must be true?
I. $x \geq 1$
II. $x \leq 2\sqrt{3}$
III. Angle $C < 60$ degrees.

(A) I only **(B)** II only **(C)** III only **(D)** I and II **(E)** I and III



(E) The triangle inequality shows that $x + 2 \leq x + (x + 1)$, so that I is true. Note that for all x with $x \geq 1$, there is a unique triangle with sides of the given length, so II is false. Finally, note that because of the lengths of the sides opposite angles $A, B,$ and C, we may conclude that $\angle C < \angle A < \angle B$. Since $\angle A + \angle B + \angle C = 180°$, we see that $3\angle C < 180°$, hence $\angle C < 60°$ and III is true.

13. If $2^{10x-1} = 1$, what is $\log x$?

(A) -1 **(B)** 0 **(C)** 1 **(D)** 2 **(E)** 3

(A) Note that $2^{10x-1} = 1$ implies $10x = 1$, $x = 0.1$ So $\log x = -1$.

14. How many positive divisors does 6! have?

(A) 4 **(B)** 6 **(C)** 10 **(D)** 20 **(E)** 30

(E) $6! = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 2^4 \cdot 3^2 \cdot 5$, so it has $(4+1)(2+1)(1+1) = 30$ positive integer divisors.

15. The vertices of a triangle T are $(0,0)$, $(0,y)$, and $(x,0)$, where x and y are positive. The area of T is 30 and the perimeter is also 30. What is $x+y$?

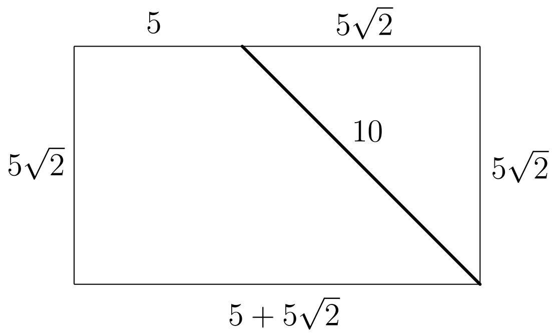
(A) 12 **(B)** 13 **(C)** 15 **(D)** 17 **(E)** 18

(D) The two unknowns x and y satisfy $xy = 60$ and $x + y + \sqrt{x^2+y^2} = 30$. Trying a few pythagorean triples, we see that $5, 12, 13$ works just right. So the sum $x + y = 5 + 12 = 17$. Alternatively, square both sides of $x + y - 30 = \sqrt{x^2+y^2}$ and eliminate to get $120 - 60(x+y) + 900 = 0$ from which it follows that $x + y = 17$.

16. An isosceles right triangular region of area 25 is cut from a corner of a rectangular region with sides of length $5\sqrt{2}$ and $5(1+\sqrt{2})$. What is the perimeter of the resulting trapezoid?

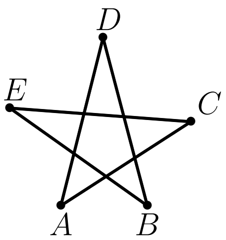
(A) 25 **(B)** 35 **(C)** $20 + 10\sqrt{2}$ **(D)** $10 + 20\sqrt{2}$ **(E)** $15 + 15\sqrt{2}$

(C) Notice from the diagram that all the sides of the trapezoid are easily determined. Adding them gives $20 + 10\sqrt{2}$.

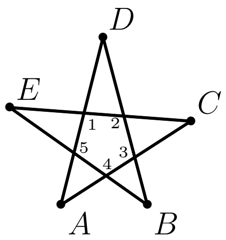


17. In the following figure, what is the sum $m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) + m(\angle E)$ of the measures of the angles A, B, C, D, and E?

(A) $180°$ **(B)** $360°$ **(C)** $540°$ **(D)** $720°$ **(E)** $900°$

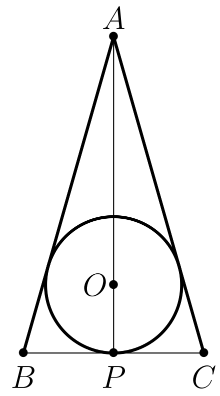


(A) Consider angles 1-5 in the center pentagon.



Because the sum of the angles of a triangle is 180°, it follows that
$m(\angle A) + m(\angle D) + m(\angle 3) = 180°$,
$m(\angle A) + m(\angle C) + m(\angle 1) = 180°$,
$m(\angle E) + m(\angle C) + m(\angle 4) = 180°$,
$m(\angle E) + m(\angle B) + m(\angle 2) = 180°$,
and $m(\angle D) + m(\angle B) + m(\angle 5) = 180°$. Adding these equations yields $2\,(m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) + m(\angle E) + (m(\angle 1) + m(\angle 2) + m(\angle 3) + m(\angle 4) + m(\angle 5) = 900°$ But since the measures of the angles of the central pentagon sum to 540°, we have $2(m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) + m(\angle E) + 540 = 900$ and $2(m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) + m(\angle E)) = 360°$. Thus $m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) + m(\angle E) = 180°$.

18. In the figure below, the circle is inscribed in an isosceles triangle ABC, with segment $\overline{AP}$ passing through the center O of the circle, $AC = AB = 12$ and $BP = 4$. Find the radius r of the circle.

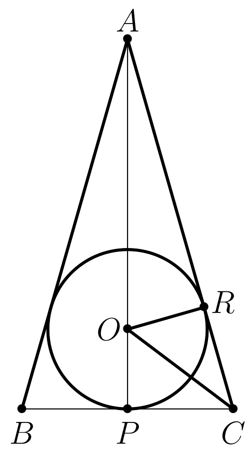


(A) $2\sqrt{2}$ **(B)** $3\sqrt{2}$ **(C)** $4\sqrt{2}$ **(D)** $6\sqrt{2}$ **(E)** $12+\sqrt{2}$

(A) See the diagram below. Since the circle is inscribed within the triangle and $\overline{AP}$ passes through the circle's center, it follows that $\overline{AP}$ is perpendicular to $\overline{BC}$. By the Pythagorean Theorem, we have $AP^2+4^2 = 12^2$ and $AP = 8\sqrt{2}$. Let R be the point of tangency of the circle with $\overline{AC}$. Consider triangles OPC and ORC. They are right triangles that have the same hypotenuse. Since $OP = OR$ is the radius of the circle, they are congruent. Hence, $PC = RC = 4$ and $AR = 12 - 4 = 8$. And in right triangle AOR, we have

$$(8\sqrt{2} - r)^2 = 8^2 + r^2.$$

Solve the equation $128 - 2 \cdot 8\sqrt{2}r + r^2 = 64 + r^2$ for the radius r to get $r = \frac{4}{\sqrt{2}} = 2\sqrt{2}$.



19. Bill and his dog walk home from the shopping center. It takes Bill 36 minutes and his dog walks twice as fast. They start together, but the dog reaches home before Bill and returns to meet Bill. After meeting Bill, the dog walks home, again at double speed, and then turns back to meet Bill again. Bill starts at noon to walk home. How many minutes later does he meet the dog for the second time?

(A) 24 **(B)** 27 **(C)** 30 **(D)** 32 **(E)** 34

(D) The dog is at home for the first time after 18 minutes. Then he runs back to Bill at double speed. It takes $\frac{18}{3} = 6$ minutes for the dog to run back towards Bill. Hence Bill and the dog meet the first time after $18 + 6 = 24$ minutes. Since $24 = \frac{2}{3}36$, at that moment, Bill has still a third of his way to walk, and the dog begins its second leg. The second leg of the dog, forward to home and backward to Bill takes $\frac{1}{3}$ of the time of the first leg, hence $\frac{24}{3} = 8$ minutes. Thus Bill and the dog meet the second time after $24 + 8 = 32$ minutes.

20. How many ordered triples (x, y, z) satisfy the equation

$$3x^2 + 3y^2 + z^2 - 2xy + 2yz = 0 \ ?$$

(A) 0 **(B)** 1 **(C)** 3 **(D)** 4 **(E)** infintely many

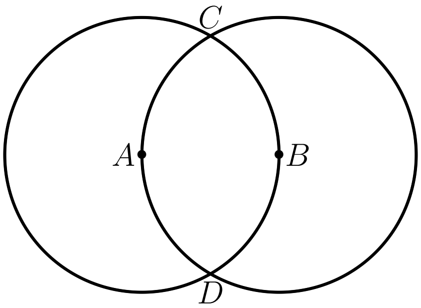
(B) We can express the left side as a sum of squares:

$$\begin{aligned} 3x^2 + 3y^2 + z^2 - 2xy + 2yz &= \\ x^2 - 2xy + y^2 + 2x^2 + y^2 + y^2 + 2yz + z^2 &= \\ (x-y)^2 + 2x^2 + y^2 + (y+z)^2. \end{aligned}$$

A sum of squares can be zero only if each one is zero. Thus $x = 0, y = 0, z = 0$ is the only solution.

21. Let A, B be the centers of two circles with radii 1 and assume that $AB = 1$. Find the area of the region enclosed by arcs CAD and CBD.

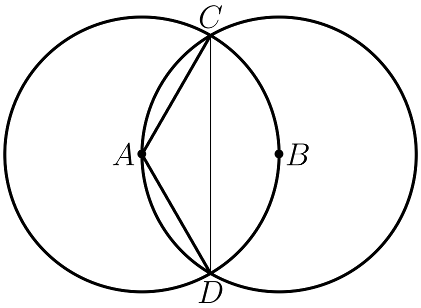
(A) 1 **(B)** $\frac{2\pi}{3}$ **(C)** $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$ **(D)** $\frac{\sqrt{3}}{2}$ **(E)** $\sqrt{3}$



(C) Note that $\angle CAD$ is $\frac{2\pi}{3}$. This implies that the area is twice the difference between area of the area of the sector CAD and the triangle CAD. In other words, it is

$$2\left(\pi \cdot 1^2/3 - \frac{1}{2}\sqrt{3} \cdot \frac{1}{2}\right)$$

which is $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$.



22. Suppose x satisfies $|x^2 - 2x - 3| = |x^2 - 2x + 5|$. Then x belongs to

(A) $[0, 2)$ **(B)** $[2, 4)$ **(C)** $[4, 6)$ **(D)** $[6, 8)$ **(E)** $[8, \infty)$

(A) Since $x^2 - 2x + 5 = (x - 1)^2 + 4$ we may omit the absolute value sign on the right hand side. Now $x^2 - 2x - 3 = (x + 1)(x - 3)$ is negative for $-1 < x < 3$, elsewhere it is not negative. Hence the equation is of the form $x^2 - 2x - 3 = x^2 - 2x + 5$, i.e, $-3 = 5$ when $x \le -1$ or $x \ge 3$. In this case we have no solution. If $-1 < x < 3$ then the equation $-x^2 + 2x + 3 = x^2 - 2x + 5$ is equivalent to $2(x - 1)^2 = 0$. Hence $x = 1$, and this is a solution since $-1 < 1 < 3$.

Alternatively, since $|x^2 - 2x - 3| = |x^2 - 2x + 5|$, it must be the case that either $(x^2 - 2x - 3) = (x^2 - 2x + 5)$ or $(x^2 - 2x - 3) = -(x^2 - 2x + 5)$. The first has no solutions, and the second is equivalent to $2x^2 - 4x + 2 = 0$ which has just one solution, $x = 1$.

23. When $\sqrt{4 - 2\sqrt{3}}$ is expressed in the form $\sqrt{a} - b$, where a and b are integers, the value of $a + b$ is

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

(C) Note that

$$\begin{aligned}\sqrt{4 - 2\sqrt{3}} &= \sqrt{(\sqrt{3})^2 - 2\sqrt{3} + 1^2} \\ &= \sqrt{(\sqrt{3} - 1)^2} \\ &= \sqrt{3} - 1,\end{aligned}$$

so $a = 3$ and $b = 1$.

Alternatively, solve $\sqrt{4 - 2\sqrt{3}} = \sqrt{a} - b$ by squaring both sides to get $4 - 2\sqrt{3} = a + b^2 - 2b\sqrt{a}$. Then it is easy to guess that $a = 3$ and $b = 1$, and check these values.

24. The sum of the reciprocals of four different positive integers is 1.85. Which of the following could be the sum of the four integers?

(A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 19

(C) Because $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5} < 1.85$, one of the integers must be 1. Let a, b and c denote the others. Since $\frac{1}{3}+\frac{1}{4}+\frac{1}{5} = \frac{47}{60} < 0.85$, one of the numbers, say a, must be 2. Then $\frac{1}{b}+\frac{1}{c} = 0.35$. Since $0.35 - 1/3 = 5/300 = 1/60$, one set of integers that works is $\{1, 2, 3, 60\}$. But $1+2+3+60 = 66$ is not one of the options. If $b = 4$, then $c = (.35 - .25)^{-1} = 10$ also works, and $1+2+4+10 = 17$ is one of the options. Note that these are the only solutions since $\frac{1}{5}+\frac{1}{6} \neq 0.35$ and all other possible sums are less than 0.35.

25. What is the area of a triangle whose sides are $5, 6$, and $\sqrt{13}$?

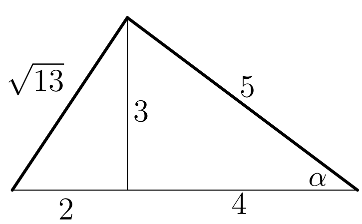
(A) $5\sqrt{2}$ **(B)** 8 **(C)** 9 **(D)** $6\sqrt{2}$ **(E)** 10

(C) Use the law of cosines to get

$$(\sqrt{13})^2 = 5^2 + 6^2 - 2(5)(6)\cos\alpha.$$

Solve for $\cos\alpha$ to get $\cos\alpha = \frac{25+36-13}{60} = \frac{4}{5}$. Then note that $\sin\alpha = 3/5$, which means the altitude to the base of length 6 is 3, so the area is 9.

Alternatively, build such a triangle from triangles we know more about. Start with a $3; 4; 5$ triangle. Note that $\sqrt{13} = \sqrt{2^2+3^2}$ so we might try to build the triangle in the problem from a $2; 3; \sqrt{13}$ and a $3; 4; 5$. We can append them along the edge of length 3. Notice that since both triangles are right, we can append them so that the union is a triangular region with sides of length $4+2, 5$, and $\sqrt{13}$, whose area is easy to find. The altitude to the base 6 is 3, so the area is 9. Yet another approach would be to use Heron's formula.



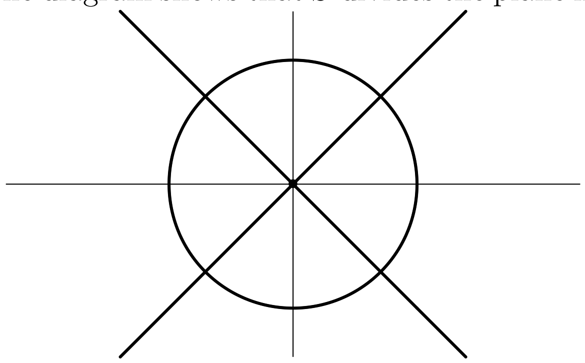
26. Into how many regions does the solution set S of

$$xy(y-x)(y+x)(x^2+y^2-1)=0$$

divide the plane? Note that some of the regions are bounded (surrounded by points of S), and some are unbounded.

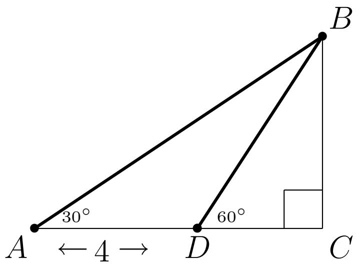
(A) 4 **(B)** 8 **(C)** 16 **(D)** 18 **(E)** 22

(C) The Zero Product Property applies. Thus one or more of the five factors is zero. Hence S consists of all points satisfying any of the following five equations: $x=0, y=0, y=-x, y=x, x^2+y^2=1$. The first four are lines and the fifth is a circle. The diagram shows that S divides the plane into 16 regions.

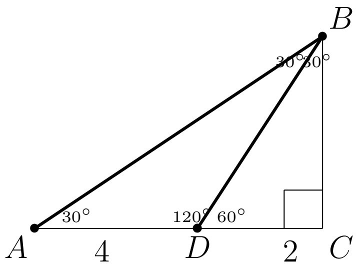


27. In the right triangle ABC shown, D is on $\overline{AC}$, $\angle A = 30°$, $\angle BDC = 60°$, and $AD = 4$. Find BC

(A) 3 **(B)** $2\sqrt{3}$ **(C)** $\sqrt{14}$ **(D)** 4 **(E)** $3\sqrt{2}$



(B) Refer to the diagram. Since $\angle ABD = \angle DAB$, it follows that $BD = 4$. Since $\triangle BCD$ is a $30, 60, 90$ triangle, $DC = BD/2 = 2$. Then $BC^2 = 4^2 - 2^2 = 12$, so $BC = 2\sqrt{3}$.



28. An integer between 100,000 and 199,999 becomes three times as big when we move the 1 from the leftmost position to the rightmost position. Find the sum of the digits of the number

(A) 22 **(B)** 24 **(C)** 27 **(D)** 28 **(E)** 29

(C) Let N denote the five-digit number obtained by subtracting 100000 from the six-digit number in the problem. Then $3(100000 + N) = 10N + 1$, which is equivalent to $7N = 299999$ which yields $N = 42857$, so the sum of the digits of the six-digit number is $1 + 4 + 2 + 8 + 5 + 7 = 27$.