Multifactor Mean-Reverting Models

1. a) Calculate the time series for 5 spreads between Canadian and USA interest rates
b) Assume the spreads follow a general linear mean-reverting system
   \[ d\bar{x}(t) = \mathbf{K} (\bar{\theta} - \bar{x}(t)) \, dt + A \, d\bar{w}, \quad \bar{x}(0) = \bar{x}_0, \quad A A^T = R \]

   Estimate the following parameters for this system from the time series of spreads: mean vector \( \bar{\theta} \);
   annualized (multiplied by 250 business days in a year) covariance matrix of the spread daily absolute returns
   \( R \) (absolute returns are \( \bar{r}_i = \bar{x}_i - \bar{x}_{i-1} \) where \( \bar{x}_i \) is a vector of spreads for day \( i \) ) and matrix \( \tilde{B} = HS \)
   with its Principal Components from the eigenvalue decomposition
   \[ \tilde{B} = HS, \quad R = (HS)(SH'), \quad H' = H^{-1}, \quad S = \text{diag}(s_1, \ldots, s_N), \quad s_1 \geq \ldots \geq s_N > 0; \]
   and, finally, the covariance matrix \( Q_{\infty} \) of spreads themselves. Plot all these parameters.

c) Estimate a full 5x5 matrix \( \mathbf{K} \) of mean-reversion speeds based on
   \[ e^{-\mathbf{K} \delta} = A_\delta Q_{\infty}^{-1}, \] where a time lag \( \delta > 0 \) (measured in years) should be chosen by students as
   \( \delta = 1/12 \times \{1+[(\text{Your student ID}) \mod 25]\} \), and autocovariance matrix \( A_\delta \) is calculated as
   \[ A_\delta = E\{(\bar{x}(t) - \bar{\theta})' (\bar{x}(t - \delta) - \bar{\theta})\} \]
   Plot matrix \( \mathbf{K} \).

d) Estimate a full 5x5 matrix \( \mathbf{K} \) of mean-reversion speeds
   - calculate the matrix \( \tilde{Q}_{\infty} = \tilde{B}^{-1} Q_{\infty} (\tilde{B}^{-1})' \) and find matrices \( V \) and \( D \) from eigenvalue decomposition
   \[ \tilde{Q}_{\infty} = VDV', \quad V' = V^{-1}, \quad D = \text{diag}(d_1, \ldots, d_N), \quad d_i > 0. \]
   Set a new diagonal matrix \( \Omega = 0.5 D^{-1} \) and calculate the original matrix \( \mathbf{K} \) of mean-reversion speeds as
   \[ \mathbf{K} = \tilde{B} V \Omega V' \tilde{B}^{-1}. \]
   Plot matrix \( \mathbf{K} \) and compare with the matrix from c).
2. Curve Simulation

All the calculations above in d) correspond to the linear change of variable $\bar{x}(t) = B \bar{y}(t) = \tilde{B} V \bar{y}(t)$ in the original mean-reverting system and the assumption that a vector of drivers $\bar{y}(t)$ follows a "diagonalizable" mean-reverting system $d\bar{y}(t) = \Omega (\bar{\mu} - \bar{y}(t)) dt + d\bar{w}, \quad \bar{y}(0) = V' \tilde{B}^{-1} \bar{x}(0), \quad \bar{\mu} = V' \tilde{B}^{-1} \bar{\theta}.$

Simulate and plot a five year path for the Canadian – US interest rate spreads with the daily time step based on the above equations. Each independent driver $y_j(t)$ simulate from an exact method:

$$y_j(t + \Delta t) = \mu_j + (y_j(t) - \mu_j) \exp(-\sigma_j \Delta t) + \sqrt{\frac{1 - \exp(-2\sigma_j \Delta t)}{2\sigma_j}} z, \quad z \sim N(0,1).$$