1. How many types of improper integrals are there? Give an example of each type.

   \[ \text{2 types: 1 with } \infty \text{ & 1 with } 0 \]

   \[
   \text{type I: example } \int_{\infty}^{1} e^x \, dx
   \]

   \[
   \text{type II: example } \int_{0}^{1} \frac{1}{x} \, dx
   \]

2. State the Trapezoid Rule, the Midpoint Rule, and Simpson’s Rule. Which of these three is the most accurate?

   \[
   T_n = \frac{\Delta x}{2} \left[ f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n) \right]
   \]

   \[
   M_n = \Delta x \left[ f(\bar{x}_1) + f(\bar{x}_2) + \ldots + f(\bar{x}_n) \right]
   \]

   \[
   S_n = \frac{\Delta x}{3} \left[ f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right]
   \]

   \[ S_n \text{ is most accurate.} \]

3. What is the general formula for the arc length of a function?

   \[
   L = \int_{a}^{b} \sqrt{(x')^2 + (y')^2} \, dt
   \]

4. What is the formula for the average value of a function? Can we always find an x-value where for that x-value such that \( f(x) = f_{\text{AVG}} \)?

   \[
   f(c) = f_{\text{AVG}} = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx
   \]

   \[ \text{yes, we can always find } c \]

5. What are all the important formulas you need to know when finding a centroid?

   \[
   M_x = \rho \int_{a}^{b} \frac{1}{2} [x^2 - g(x)] \, dx \quad \text{or} \quad M_x = \rho \int_{a}^{b} x \left[ f(x) - g(x) \right] \, dx
   \]

   \[
   M_y = \rho \int_{a}^{b} y \left[ f(x) - g(x) \right] \, dx
   \]

   \[
   m = \rho A
   \]

   \[
   \bar{x} = \frac{M_y}{m}
   \]

   \[
   \bar{y} = \frac{M_x}{m}
   \]
6. Calculate the following integrals:

a. \[ \int_{-\infty}^{\infty} \frac{1}{x^2+1} \, dx = \arctan(x) \bigg|_{-\infty}^{\infty} = \lim_{x \to \infty} \arctan(x) - \lim_{x \to -\infty} \arctan(x) = \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) = \pi \]

b. \[ \int_{1}^{\infty} e^{-5x} \, dx = -\frac{1}{5} e^{-5x} \bigg|_{1}^{\infty} = \lim_{t \to \infty} \left[ -\frac{1}{5} e^{-5t} + \frac{1}{5} e^{-5} \right] = \frac{1}{5} e^{-5} \]

c. \[ \int_{0}^{5} \frac{3}{2x-5} \, dx = \int_{0}^{5/2} \frac{3}{2x-5} \, dx + \int_{5/2}^{5} \frac{3}{2x-5} \, dx \]
\[ = \lim_{t \to 5/2} \left[ \frac{3}{2} \ln |2x-5| \right]_{0}^{5/2} = \lim_{t \to 5/2} \left[ \frac{3}{2} \ln |2x-5| - \frac{3}{2} \ln 5 \right] = -\infty \]
\[ \text{diverges} \]

d. \[ \int_{0}^{1/4} \frac{4}{x^2} \, dx = \lim_{t \to 0} \int_{t}^{1/4} \frac{4}{x^2} \, dx = \lim_{t \to 0} \left[ -\frac{4}{x} \right]_{t}^{1/4} = \lim_{t \to 0} \left[ -4 + \frac{4}{4} \right] = \infty \]
\[ \text{diverges} \]

7. Calculate the following using numerical integration: \[ \int_{0}^{\sqrt{\frac{\pi}{4}}} \sin(x^2) \, dx \]

a. Which of the following is the error estimate for the Trapezoidal Rule?

\[ \frac{K(b-a)^3}{12n^2} \]
\[ \frac{K(b-a)^3}{24n^2} \]
\[ \frac{K(b-a)^5}{180n^4} \]

b. If we want our answer to be accurate to within 0.01 of the true solution, what should our \( n \) be for the trapezoidal rule?

\[ 2 \left( \frac{\sqrt{\frac{\pi}{4}}}{n} \right)^3 \leq 0.01 \]
\[ = \frac{1.4500854}{n^2} \leq 0.01 \]
\[ n^2 = 1204 \leq n \]
\[ \boxed{n = 2} \]

(c) Using the \( n \) you found in part b, find \( T_n \)

\[ T_2 = \frac{\Delta x}{2} \left[ f(x_0) + 2f(x_1) + f(x_2) \right] \]
\[ = \frac{\sqrt{\frac{\pi}{4}}}{12} \left[ \sin 0 + 2\sin \left( \frac{4\pi}{4} \right) + \sin \left( \frac{5\pi}{4} \right) \right] \]
\[ = 0.032483060 \]

(d) Find the true solution of the integral (either by hand or using your calculator).

\[ \int_{0}^{\sqrt{\frac{\pi}{4}}} \sin(x^2) \, dx = 0.28921941436 \]

(e) What is the actual error of the \( T_n \) you found in part c.

\[ E_T = -0.00356111856 \]
8. Find the area between the following curves. Draw a picture of the situation, decide whether to integrate with respect to \( x \) or \( y \), and make sure to find the correct bounds of the figure. Be careful some of these are trick questions.

a. Find the area between \( y = 4x + 1 \) and \( y = e^x \) and \( x = 2 \)

\[
\begin{align*}
\int_0^2 4x + 1 - e^x \, dx &= 2x^2 + x - e^x \bigg|_0^2 \\
&= 10 - (e^2 - 1) \\
&= 11 - e^2 \approx 3.6109
\end{align*}
\]

b. Find the area between \( y = 3x \) and \( y^2 + 8x = 1 \)

\[
x = \frac{1}{3} \, y \\
x = (1 - y^2) \cdot \frac{1}{8} \quad , \quad -3 \leq y \leq \frac{1}{3}
\]

\[
\int_{-3}^{1/3} \frac{1}{8} (1 - y^2) - \frac{1}{3} y \, dy \\
= -0.7714049383
\]

c. Find the area between \( y = \sin x \) and \( y = 2 \sin x \) on the interval \([0, 2\pi]\)

\[
\begin{align*}
\int_0^{2\pi} 2 \sin x - \sin x \, dx + \int_0^{2\pi} \sin x - 2 \sin x \, dx \\
= \int_0^{2\pi} \sin x \, dx - \frac{\sin x}{2} \bigg|_0^{2\pi} \\
&= -\cos x \bigg|_0^{2\pi} \\
&= 2
\end{align*}
\]

d. Find the area enclosed by \( y = x^2 \) and \( y = e^x + 1 \) and \( y = -x + 2 \)

\[
\begin{align*}
\int_{-1.4147758}^0 e^x + 1 - x^2 \, dx + \int_0^1 -x + 2 - x^2 \, dx \\
= 1.324411469 + 1.16 \\
= 2.493078136
\end{align*}
\]
9. Suppose each of the given regions is rotated about the given axis of rotation. Find the volume of the resulting figures.

   a. The region defined by the area enclosed by the functions \( y = x \) and \( y = x/4 \) is rotated about the x-axis.

\[
\begin{align*}
V &= \int_0^1 \pi \left( R^2 - r^2 \right) \, dx \\
&= \pi \int_0^1 x^2 - \frac{x^2}{16} \, dx \\
&= \frac{15\pi}{16} \int_0^1 x^2 \, dx \\
&= \frac{15\pi}{16} \cdot \frac{1}{3} = \frac{5\pi}{16}
\end{align*}
\]

   b. The region defined by the area enclosed by the functions \( y = x \) and \( y = 10x^4 \) is rotated about \( x = -2 \).

\[
\begin{align*}
V &= \int_0^{4/4} \pi \left( \left[ 2 + \left( \frac{y}{10} \right)^{4/4} \right]^2 - \left[ 2 + y \right]^2 \right) \, dy \\
&= 0.9169243963
\end{align*}
\]

   c. The region is defined by the area enclosed by the function \( y = x(x-4) \) and is rotated about the y-axis.

\[
\begin{align*}
V &= \int_0^4 2\pi x^2 (x+4) \, dx \\
&= 2\pi \int_0^4 x^3 + 4x^2 \, dx \\
&= \frac{128\pi}{3}
\end{align*}
\]

Cylindrical:
\[
2\pi x [x(x-4)]
\]
10. Consider the straight line $y = 2x + 1$
   a. What is the arc length of this function on the interval $[0, 2]$?
      \[ L = \int_0^2 \sqrt{(x')^2 + (y')^2} \, dx = \int_0^2 \sqrt{1 + (2)^2} \, dx = \int_0^2 \sqrt{5} \, dx = \sqrt{5} x \bigg|_0^2 = 2\sqrt{5} \]
   
      b. Use the distance formula \[ d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \] to find the distance between the points $(0, 1)$ and $(2, 5)$.
      \[ d = \sqrt{(0 - 2)^2 + (1 - 5)^2} = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5} \]
   
   c. Do you get the answer you expect? Why or why not?
      yes, because the distance formula measures straight line distance & \[ y = 2x + 1 \] is a straight line

11. Consider the parametric function: $x(t) = 5 \cos t$ and $y(t) = 5 \sin t$
   a. What is the arc length of this function on $0 \leq t \leq \pi$ (keep your answer in terms of $\pi$)?
      \[ L = \int_0^\pi \sqrt{(x')^2 + (y')^2} \, dt = \int_0^\pi \sqrt{(-5 \sin t)^2 + (5 \cos t)^2} \, dt \]
      \[ = \int_0^\pi \sqrt{25} \, dt = \sqrt{25} t \bigg|_0^\pi = 5\pi \]
   
   b. Graph this function. (You should get a circle.)
      
      \[ \text{Graph of } x = 5 \cos t, y = 5 \sin t \]

   c. Use the formula for the circumference of a circle ($C = 2\pi r$) to find the circumference of one half of a circle with radius 5. Do you get the answer you expect?
      \[ \frac{1}{2} C = \frac{1}{2} \pi \cdot 2r = \pi r \]
      \[ \text{if } r = 5 \]
      \[ \Rightarrow \text{ half circumference } = 5\pi \]
12. Consider the function \( f(x) = (x + 2)^2 \) on the interval \([0,9]\)
   a. Find the Mean value of \( f(x) \)
      \[
      f_{\text{avg}} = \frac{1}{9-0} \int_0^9 (x+2)^2 \, dx = 49
      \]
   b. Find all values \( c \) where \( 0 \leq c \leq 9 \) and \( f(c) = f_{\text{avg}} \)
      \[
      f(c) = 49
      \]
      \[
      (x+2)^2 = 49
      \]
      \[
      x + 2 = \pm 7 \Rightarrow \]
      \[
      x = -2 \pm 7 \Rightarrow c = -9, 5
      \]

13. A trough has two ends which are isosceles trapezoids with bases 6 meters and 4 meters, a height of 5 meters and a length of 8 meters (see diagram). Suppose this trough is filled with water (with density 1000 kg/m\(^3\)). Find the work required to pump the water out of the tank, leaving a depth of 1 meter at the bottom of the trough.

\[
\Delta W = \Delta F \cdot \text{d} \theta t
\]
\[
\text{d} \theta t = 5 - y \quad \# \quad 1 \leq y \leq 5
\]
\[
\Delta F = (\Delta \text{mass})(\text{gravity})
\]
\[
\Delta \text{mass} = (\Delta \text{volume})(\text{density})
\]
\[
\Delta \text{volume} = 8 \cdot 2x \cdot \Delta y = 16 \left( \frac{y+10}{5} \right) \Delta y = \left( \frac{16}{5} y + 32 \right) \Delta y
\]
\[
\Delta \text{mass} = (3200 y + 32000) \Delta y
\]
\[
\Delta F = (31360 y + 313600) \Delta y, \quad \eta = 9.8
\]
\[
\Rightarrow \Delta W = (31360 y + 313600)(5-y) \Delta y
\]
\[
W = \int_1^5 (31360 y + 313600)(5-y) \, dy
\]
\[
= 3094186.667 \text{ Joules}
\]
14. A force of 25 pounds is required to compress a spring 6 inches from its natural length of 30 inches. Find the work done in compressing the spring an additional 6 inches.

\[ \int_{\frac{1}{2}}^{1} 50x \, dx = 25x^2 \bigg|_{\frac{1}{2}}^{1} = 18.75 \text{ ft-lb} \]

15. For each of the following situations, find the moments & the Centers of Mass (Centroids)

a. \((0, 5), (1, -4), (2, 3), (-1, 0)\) with the associated masses 3, 5, -3, 1

\[ m_x = 3(5) + 5(-4) + (-3)(3) + 1(0) = -14 \]
\[ m_y = 3(0) + 5(1) + (-3)(2) + 1(-1) = -2 \]
\[ m = 3 + 5 - 3 + 1 = 6 \]
\[ \bar{x} = \frac{-7}{6} \bar{y} = \frac{-2}{6} \implies (\bar{x}, \bar{y}) = (\frac{-1}{3}, \frac{-1}{3}) \]

b. The region between the curve \(y = -2x(x-2)\) and the x-axis

\[ m_x = \rho \int_{0}^{2} \frac{1}{2} (\text{A}) \, dy = \frac{32}{15} \rho \]
\[ m_y = \rho \int_{0}^{2} x (\text{A}) \, dy = \frac{8}{3} \rho \]
\[ m = \rho A = \frac{8}{3} \rho \]
\[ \bar{x} = \frac{m_y}{m} = 1 \]
\[ \bar{y} = \frac{m_y}{m} = \frac{\frac{32}{15}}{\frac{8}{3}} = \frac{8}{5} \]

Centroid: \((1, \frac{4}{5})\)

![Graph of y = -2x(x-2)](image)

A = \int_{0}^{2} -2x(x-2) \, dx = 2.67 \approx 8/3

f(x) = -2x(x-2) \quad g(x) = 0

\[ A = \int_{0}^{2} -2x(x-2) \, dx = 2.67 \approx 8/3 \]

\[ A = \int_{0}^{2} x-x^3 \, dx \]

\[ A = \frac{1}{4} \]

C. The region between the two curves \(y = x\) and \(y = x^3\) where \(x \geq 0\)

\[ m_x = \rho \int_{0}^{1} \frac{1}{2} \left( x^2 - x^6 \right) \, dx = \frac{2\rho}{21} \]
\[ m_y = \rho \int_{0}^{1} x \left( x - x^3 \right) \, dx = \frac{2\rho}{15} \]
\[ m = \rho A = \rho/4 \]
\[ \bar{x} = \frac{m_y}{m} = \frac{\frac{2\rho}{15}}{\frac{\rho}{4}} = \frac{8}{15} \]
\[ \bar{y} = \frac{m_x}{m} = \frac{\frac{2\rho}{21}}{\frac{\rho}{4}} = \frac{8}{21} \]

Centroid: \((\frac{8}{15}, \frac{8}{21})\)