Sketch a graph of the given curve and find its exact length using the Arc Length formula.

1. \( x = 2 + t^3, \; y = 1 + 2t^2, \; 0 \leq t \leq 2. \)

\[
L = \int_0^2 \sqrt{(3t^2)^2 + (4t)^2} \, dt \\
= \int_0^2 \sqrt{9t^4 + 16} \, dt \\
= \frac{1}{6} \left[ \frac{2}{3} (9t^2 + 16)^{3/2} \right]_0^2 \\
= \frac{1}{6} \left[ 52^{3/2} - 64 \right] = 11.51767899
\]

2. \( x = e^{t} - t, \; y = 4e^{t/2}, \; -8 \leq t \leq 3. \)

\[
L = \int_{-8}^{3} \sqrt{(e^t - 1)^2 + (2e^{t/2})^2} \, dt \\
= \int_{-8}^{3} \sqrt{e^{2t} - 2e^t + 1 + 4e^t} \, dt \\
= \int_{-8}^{3} \sqrt{e^{2t} + 2e^t + 1} \, dt \\
= \int_{-8}^{3} e^t + 1 \, dt \\
= e^t + t \bigg|_{-8}^{3} = e^3 + 3 - e^{-8} + 8 \\
= e^3 - e^{-8} + 11
\]

3. \( x = 7 \cos t, \; y = 3 \sin t, \; 0 \leq t \leq 2\pi. \)

\[
x' = -7\sin t, \; y' = 3 \cos t \\
L = \int_0^{2\pi} \sqrt{(-7\sin t)^2 + (3\cos t)^2} \, dt \\
= \int_0^{2\pi} \sqrt{49\sin^2 t + 9\cos^2 t} \, dt \\
= \int_0^{2\pi} \sqrt{49 + 9} \, dt \\
= \int_0^{2\pi} \sqrt{58} \, dt \\
= 32.68546675
\]

4. \( x = y^{3/2}, \; 0 \leq y \leq 1. \)

\[
x' = \frac{3}{2} y^{1/2} \\
L = \int_0^1 \sqrt{1 + \left(\frac{3}{2} y^{1/2}\right)^2} \, dy \\
= \int_0^1 \sqrt{1 + \frac{9}{4} y} \, dy \\
= \frac{4}{3} \int \sqrt{u} \, du \\
= \frac{4}{3} \left[ \frac{2}{3} u^{3/2} \right]_0^1 \\
= \frac{8}{27} \left[ 1 + \frac{9}{4} \right]^{3/2} \\
= \frac{8}{27} \left[ \frac{13}{4} \right]^{3/2} \\
= 1.439709873
\]
5. Consider: \( x = r \cos t, \ y = r \sin t, \ 0 \leq t \leq 2\pi, \) where \( r \) is any positive real number.

a. Sketch the graph of the curve.

The graph is a \textcircled{c}ircle with radius \( r \)

b. Find the arc length of this figure. (Leave your answer in terms of \( \pi \).)

\[ x' = -rsin t \quad \text{and} \quad y' = rcos t \]

The length is \( L = \int_{0}^{2\pi} \sqrt{(-rsint)^2 + (rcost)^2} \) \( dt \)

\[ \begin{align*}
&= r \int_{0}^{2\pi} \sqrt{\sin^2 t + \cos^2 t} \) \( dt \) \quad \text{by factoring} \ r \text{ out} \\
&= r \int_{0}^{2\pi} 1 \) \( dt \) \quad \text{\( \int_{0}^{2\pi} 1 \) \( dt \) \quad \text{\( r \) \( \int_{0}^{2\pi} \) \( dt \) \quad \text{\( r \) \( 2\pi - 0 \) \( r \) \( 2\pi r \)}} \\
\end{align*} \]

c. Is this the expected answer? \( \text{yes; since arc length of entire circle is the same as circumference} \)

6. Find the average value of \( g(x) = x^2 \sqrt{1 + x^3} \) on the interval \([0, 2]\).

\[ \frac{1}{2-0} \int_{0}^{2} x^2 \sqrt{1 + x^3} \) \( dx \) = \frac{1}{2} \int_{0}^{2} x^2 \sqrt{1 + x^3} \) \( dx \)

\[ = \frac{1}{2} \cdot \frac{2}{3} \int \sqrt{u} \) \( du \) = \left( \frac{2}{3} \right) \frac{2}{3} u^{3/2} = \frac{1}{3} u^{3/2} \]

\[ = \frac{1}{3} (1 + x^3)^{3/2} \) \( \bigg|_{0}^{2} \) \( = \frac{1}{3} \left[ (2^3 - 1) \right] = \frac{1}{3} \left[ 8 - 1 \right] = \frac{7}{3} = 2.33 \]

7. If \( f(x) \) is continuous and \( \int_{1}^{3} f(x) \) \( dx \) = 8, show that \( f(x) \) equals 4 at least once on the interval \([1, 3]\).

\[ f_{AVG} = \frac{1}{3-1} \int_{1}^{3} f(x) \) \( dx \) = \frac{1}{2} \cdot 8 = 4 \]

by MVT, \( \Rightarrow \) There exists an \( x = c \) such that \( 1 \leq c \leq 3 \)

where \( f(c) = f_{AVG} = 4 \)

8. Find the number(s) \( b \) such that the average value of \( f(x) = 2 + 6x - 3x^2 \) on the interval \([0, b]\) is equal to 3.

\[ 3 = \frac{1}{b-0} \int_{0}^{b} 2 + 6x - 3x^2 \) \( dx \)

\[ 3 = \frac{1}{b} \left[ 2x + 3x^2 - x^3 \right] \) \( \bigg|_{0}^{b} = \frac{1}{b} \left( 2b + 3b^2 - b^3 \right) \]

\[ 3 = 2 + 3b - b^2 \]

\[ b^2 - 3b + 1 = 0 \]

\[ b = \frac{3 \pm \sqrt{9 - 4(1)(1)}}{2} = \frac{3 \pm 1}{2} \]

So \( b = \frac{3 \pm \sqrt{5}}{2} \)
9. Suppose \( f(x) = (x - 3)^2 \)
   
   a. Find the average value of \( f(x) \) on the interval \([2, 5]\)
   
   \[
   \frac{1}{5-2} \int_2^5 (x-3)^2 \, dx = \frac{1}{3} \int_2^5 x^2 - 6x + 9 \, dx = \frac{1}{3} \left( \frac{x^3}{3} - 3x^2 + 9x \right)_2^5
   \]
   
   \[
   = \frac{5^3}{3} - \frac{3^3}{3} - \frac{2^3}{3} = 3 \frac{8}{9} - 2 \frac{8}{9} = 1
   \]

   b. Find \( c \) such that \( f'_{\text{AVG}} = f''(c) \).
   
   \[
   \frac{1}{3} = f'_{\text{AVG}} = f''(c) = f'(c) = (c - 3) = c^2 - 6c + 9
   \]
   
   \[
   0 = c^2 - 6c + 9
   \]
   
   \[
   0 = (c - 3)(c - 2)
   \]
   
   \[
   c = 3, 2
   \]

   c. Sketch the graph of \( f(x) \) and a rectangle whose area is the same as the area under
   the graph of \( f(x) \).

   ![Graph of a function with a rectangle]

10. Let the equation \( h(t) = 49 - t^2 \) represent the height of an object, where \( h \) is its height in
    meters and \( t \) is the time in seconds.
   
   a. Find the time \( t = b \) when the object hits the ground (ie: solve for \( b \)).
   
   \[
   0 = 49 - t^2
   \]
   
   \[
   t^2 = 49
   \]
   
   \[
   t = \pm 7
   \]
   
   So \( b = 7 \)

   b. Find the average height of the object from time \( t = 0 \) until time \( t = b \).
   
   \[
   \frac{1}{b-0} \int_0^b 49 - t^2 \, dt = \frac{1}{7} \int_0^7 49 - t^2 \, dt
   \]
   
   \[
   = \frac{1}{7} \left[ 49t - \frac{t^3}{3} \right]_0^7 = \frac{1}{7} \left[ 49 \cdot 7 - \frac{7^3}{3} \right] = \frac{7}{3} \cdot 49 = \frac{98}{3}
   \]
   
   \[
   = 32.6
   \]

   c. Using the mean value theorem, find the time \( c \) at which the object reaches its
   average height as found in part (b).
   
   \[
   \frac{98}{3} = h'_{\text{AVG}} = h'(c) = 49 - c^2
   \]
   
   \[
   \frac{98}{3} = 49 - c^2
   \]
   
   \[
   c^2 = \frac{98}{3}
   \]
   
   \[
   c = \pm \frac{7}{\sqrt{3}}
   \]
   
   So \( c = \frac{7}{\sqrt{3}} \) or \( \frac{7\sqrt{3}}{3} \)