1. Find the area between the curves: \( y = 2x \) and \( y = x^4 \).
\[
\int_0^a 2x - x^4 \, dx = x^2 - \frac{1}{5} x^5 \bigg|_0^a
\]
\[
= a^2 - \frac{1}{5} a^5 = 0.9496539
\]

2. Find the area between the curves: \( y = 2\sin x \) and \( y = 5\cos x \) from \( x = 0 \) and \( x = \pi \).
\[
\int_0^a 5\cos x - 2\sin x \, dx + \int_a^\pi 2\sin x - 5\cos x \, dx
\]
\[
= 5\sin x + 2\cos x \bigg|_0^a + -2\cos x - 5\sin x \bigg|_a^\pi
\]
\[
= \left(\frac{5\sin(a) + 2\cos(a)}{2}\right) - \left(\frac{0 + 2}{2}\right) + \left(\frac{-2\cos(\pi) - 5\sin(\pi)}{2}\right) - \left(\frac{-2\cos(a) - 5\sin(a)}{2}\right)
\]
\[
= 10\sin(a) + 4\cos(a) = 10.77632961
\]

3. Find the volume of the region bounded by \( y = -x^3 + 5x^2 - 2x \) in the 1st quadrant, which is then rotated about the y-axis.
\[
V = \int_a^b 2\pi x (-x^3 + 5x^2 - 2x) \, dx
\]
\[
= 2\pi \int_a^b -x^4 + 5x^3 - 2x^2 \, dx
\]
\[
= 2\pi \left[-\frac{1}{5} x^5 + \frac{5}{4} x^4 - \frac{2}{3} x^3 \right]_a^b
\]
\[
= 521.147128
\]

4. Find the volume of the region bounded by \( y = 5 - x^2 \), \( y = 2x \), and the y-axis, which is rotated about \( y = -1 \).
\[
A(x) = \pi R^2 = \pi r^2
\]
\[
= \pi \left[ (6-x^2)^2 - (1+2x)^2 \right]
\]
\[
= \pi \left[ 36 - 12x^2 + x^4 - 1 - 4 - 4x^2 \right]
\]
\[
= \pi \left[ 36 - 16x^2 + x^4 \right]
\]
\[
V = \pi \int_0^a x^4 - 16x^2 - 4x + 35 \, dx
\]
\[
= \pi \left[ -\frac{1}{5} x^5 - \frac{2}{3} x^3 + 2x^2 + 35x \right]_0^a
\]
\[
= 99,172,598,22 \]
\[
R = 1 + 2x
\]
\[
r = 1 + (5 - x^2)
\]
5. Find the arc length of \( y = t^4 + 1, \quad x = -t^3 - 4, \quad 0 \leq t \leq 2 \)

\[
L = \int_a^b \sqrt{(x')^2 + (y')^2} \, dt = \int_0^2 \sqrt{(4t^3)^2 + (-3t^2)^2} \, dt
\]

\[
= \int_0^2 \sqrt{16t^6 + 9t^4} \, dt = \int_0^2 t^2 \sqrt{16t^2 + 9} \, dt
\]

\[
= 18.0193901925 \quad \text{(by calculator)}
\]

6. Find the length of the curve \( y = \frac{1}{6}(x^2 + 4)^{3/2}, \quad 0 \leq x \leq 3 \)

\[
L = \int_a^b \sqrt{(x')^2 + (y')^2} \, dx = \int_0^3 \sqrt{1 + \left( \frac{1}{2} x(x^2 + 4)^{-1/2} \right)^2} \, dx
\]

\[
= \int_0^3 \sqrt{1 + \frac{1}{4} x^2(x^2 + 4)^{-1}} \, dx
\]

\[
= 15 \frac{\sqrt{2}}{2} \quad \text{(by calculator)}
\]

7. Find the arc length of \( y = x^2, \quad 0 \leq x \leq 2 \)

\[
L = \int_a^b \sqrt{(x')^2 + (y')^2} \, dx = \int_0^2 \sqrt{1 + (2x)^2} \, dx
\]

\[
= \int_0^2 \sqrt{1 + 4x^2} \, dx
\]

\[
= 4.646878376723 \quad \text{(by calculator)}
\]

8. Find the average value of \( f(x) = x^2 \sqrt{1 + x^3} \) on \([0, 2]\)

\[
\bar{f}_{av} = \frac{1}{b-a} \int_a^b f(x) \, dx
\]

\[
= \frac{1}{2-0} \int_0^2 x^2 \sqrt{1 + x^3} \, dx
\]

\[
= \frac{1}{6} \int_0^2 \sqrt{u} \, du = \frac{1}{6} \int_1^8 \frac{u^{3/2}}{3} \, du
\]

\[
= \frac{1}{6} \left[ \frac{2}{3} u^{3/2} \right]_1^8
\]

\[
= \frac{1}{9} \left[ 2^3 - 1 \right] = 2/9
\]

\[
= \frac{2}{9} \quad \text{(by calculator)}
\]
9. Let \( f(x) = x^2 - 4x + 2 \) on \([-1, 5]\)

a) Find the average value of \( f(x) \)

\[
\bar{f}_{\text{AVE}} = \frac{1}{5-(-1)} \int_{-1}^{5} (x^2 - 4x + 2) \, dx = \frac{1}{6} \int_{-1}^{5} x^2 - 4x + 2 \, dx
\]

\[
= \frac{1}{6} \left[ \frac{x^3}{3} - 2x^2 + 2x \right]_{-1}^{5} = \frac{1}{6} \left[ 125 - (2 - 2) \right]
\]

\[
= 1
\]

b) Find \( c \) such that \( f(c) = \bar{f}_{\text{AVE}} \), where \(-1 \leq c \leq 5\)

\[
f(c) = 1
\]

\[
4c^2 - 4c + 2 = 1
\]

\[
c^2 - 4c + 1 = 0
\]

\[
c = \frac{4 \pm \sqrt{4^2 - 4(1)(2)}}{2} = 2 \pm \sqrt{2} = 2 \pm \sqrt{2}
\]

10. A force of 30 N is required to maintain a spring stretched from its natural length of 12 cm to a length of 15 cm. How much work is done in stretching the spring from 12 cm to 20 cm?

\[
f(x) = kx
\]

\[
30 = k(12), \quad k = \frac{30}{12} = 2.5
\]

\[
W = \int_{0}^{.12} 1000x \, dx = 500x^2 \bigg|_{0}^{.12}
\]

\[
= 500 (.12)^2 = 7.2 \text{ J}
\]

11. Find the centroid of \(
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{a_semicircle.png}
\end{array}
\)

(a semicircle) with \( \rho = 2 \).

\[
\bar{x} = \frac{1}{A} \int_{a}^{b} x \left( f(x) - g(x) \right) \, dx
\]

\[
= \frac{1}{2\pi} \int_{-2}^{2} x \sqrt{4 - x^2} \, dx = 0
\]

\[
\bar{y} = \frac{1}{A} \int_{a}^{b} \frac{1}{2} \left( f(x)^2 - g(x)^2 \right) \, dx
\]

\[
= \frac{1}{2\pi} \int_{-2}^{2} x^2 \, dx = \frac{8}{3}\pi
\]

\[
\text{Centroid} = (\bar{x}, \bar{y}) = (0, \frac{8}{3}\pi) = (0, 8.48526)
\]
12. A hemispherical tank (see diagram) is full of water, and has a radius of 4m. Find the work needed to pump the water out of the tank until only a height/depth of 50cm of water remains in the tank. Use: density of water is 1000 kg/m$^3$.

\[ \Delta W = (\Delta F)(\text{dist}) \]
\[ \text{dist} = 4 - y \]
\[ \Delta F = (\text{mass}) \cdot \text{gravity} \]
\[ \text{mass} = (\text{density}) \cdot (\Delta \text{Volume}) \]
\[ \Delta \text{Volume} = r^2 \pi \Delta y = (8y - y^2) \pi \Delta y \]
\[ \Delta M = (1000 \text{kg/m}^3)(8y - y^2) \Delta y \text{ kg} \]
\[ \Delta F = 1000 \pi (8y - y^2) \Delta y \text{ N} \]

\[ \Delta W = 9800 \pi [(4 - y) (8y - y^2)] \Delta y \]
\[ W = \int_{\frac{1}{2}}^{4} 9800 \pi (4 - y) (8y - y^2) \, dy = 9800 \pi \int_{\frac{1}{2}}^{4} y^3 - 12y^2 + 32y \, dy \]
\[ = \frac{4741975 \pi}{8} = 18,621,69,227 \text{ J} \]

13. A cone-shaped tank (see diagram) is full of water. Find the work needed to pump the water to a height of 2ft above the top of the tank. Use: weight of water is 62.5 lb/ft$^3$.

\[ \Delta W = (\Delta F)(\text{dist}) \]
\[ \text{dist} = 7 - y \]
\[ \Delta F = \text{weight} \]
\[ = (\text{unit weight}) \times \Delta \text{Volume} \]
\[ = (62.5 \text{lb/ft}^3) \left[ \pi r^2 \Delta y \text{ ft}^3 \right] = (62.5) \left[ \pi \left( \frac{\sqrt{2}y}{2} \right)^2 \Delta y \right] \text{ lb} \]
\[ = 10 \pi y^2 \Delta y \text{ lb} \]

\[ \Delta W = (\Delta F)(\text{dist}) = 10 \pi y^2 (7 - y) \Delta y \text{ ft-lb} \]
\[ W = \int_0^5 10 \pi (7y^2 - y^3) \, dy = 10 \pi \int_0^5 7y^2 - y^3 \, dy \]
\[ = \frac{8125 \pi}{6} = 4254.24005174 \text{ J} \]
14. Find the center of mass given the points $P_1(1, 1)$, $P_2(-3, 0)$, $P_3(2, 1)$, and $P_4(1, -2)$ with their associated weights: $m_1 = 1$, $m_2 = 5$, $m_3 = 7$, $m_4 = 2$

Draw a diagram of the situation.

\[ M_x = \sum_{i=1}^{4} m_i y_i = 1(1) + 5(0) + 7(1) + 2(-2) = 4 \]

\[ M_y = \sum_{i=1}^{4} m_i x_i = 1(1) + 5(-3) + 7(2) + 2(1) = 2 \]

\[ m = \sum_{i=1}^{4} m_i = 1 + 5 + 7 + 2 = 15 \]

\[ \bar{x} = \frac{M_y}{m} = \frac{2}{15} \]

\[ \bar{y} = \frac{M_x}{m} = \frac{4}{15} \]

Center of mass: $(\bar{x}, \bar{y}) = (\frac{2}{15}, \frac{4}{15})$