ON THE DETERMINATION OF THE BOUNDARY IMPEDANCE FROM THE FAR FIELD PATTERN∗

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Abstract. We consider the Helmholtz equation in the half space and suggest two methods for determining the boundary impedance from knowledge of the far field pattern of the time-harmonic incident wave. We introduce a potential for which the far field patterns in specially selected directions represent its Fourier coefficients. The boundary impedance is then calculated from the potential by an explicit formula or from the WKB approximation. Numerical examples are given to demonstrate the efficiency of our approaches. We also discuss the validity of the WKB approximation in determining the impedance of an obstacle.

Key words. inverse scattering problem, boundary impedance

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1. Introduction. Various impurities such as gases, dust, or cracks, on the surface of a body subject to an incident wave can be modeled by the impedance boundary condition [1]. The detection of these inhomogeneities using nondestructive testing is then reduced to the reconstruction of the impedance from measurements of the scattering field [2]. Optical scanning of the surface of silicon wafers used for quality control in the semiconductor industry [3] is one of the possible applications of this method.

We consider the scattering of an incident time-harmonic plane wave from the boundary of the half space \( \mathbb{R}_+^3 = \{ x = (x_1, x_2, x_3), x_3 > 0 \} \). The problem is described by the Helmholtz equation

\[ -\Delta u = k^2 u, \quad x_3 > 0, \]

with the impedance boundary condition

\[ u_{x_3} + ik\gamma(x')u|_{x_3=0} = 0, \quad x' = (x_1, x_2, 0), \]

where \( \gamma(x') \) is the surface impedance with a bounded support \( \text{supp} \gamma \subset [-1,1] \times [-1,1] \) and \( u = u(x) \) is the superposition of the incident, reflected, and scattered waves:

\[ u(x) = e^{ik \cdot x} + e^{ik^* \cdot x} + \psi(x). \]

Here \( k = (k_1, k_2, k_3) \) is a vector such that \( |k| = k, \ k^* = (k_1, k_2, -k_3) \), and function \( \psi(x) \) satisfies the radiation condition

\[ \psi(x) = \frac{e^{ik|x|}}{|x|} \left[ f(k, \hat{x}) + O \left( \frac{1}{|x|} \right) \right], \quad \hat{x} = \frac{x}{|x|} \in \mathbb{S}^2. \]

The inverse scattering problem for (1.1)–(1.2) consists in determining the impedance \( \gamma(x') \) by the far field pattern \( f = f(k, \hat{x}) \) when \( k \) is fixed and \( \hat{x} \in \mathbb{S}^2 \).

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In the next section we introduce a modified potential $v$ and express the impedance $\gamma$ through $v$ using an explicit formula. The mapping $v \to f$ is linear. Hence, the initial nonlinear inverse problem is split into two steps: solution of a linear problem (restoring $v$ from $f$) and application of the explicit formula. A similar approach was used in the discrete counterpart of the problem [4]. This approach does not formally require $k \gg 1$. We also modify it for large $k$ using the WKB method.

In the case of a bounded obstacle, the WKB method allows one to connect the asymptotic solution of (2.3) has the form

\begin{align}
\gamma &= c(x)k^2 u, \quad x \in \mathbb{R}^3,
\end{align}

where potential $q(x) = -2ik\gamma(x')\delta(x_3)$ and $\delta(x)$ is the Dirac delta-function.

Substituting (1.3) into (2.1), we obtain that the scattering solution $\psi(x)$ satisfies

\begin{align}
(-\Delta + q(x) - k^2)\psi = -q(x)(e^{ik\cdot x} + e^{ik\cdot -x}) = -2q(x)e^{ik\cdot x'}, \quad k' = (k_1, k_2, 0).
\end{align}

Equation (2.2) is uniquely solvable if $\psi$ satisfies the radiation conditions (1.4). From (2.2) it follows that

\begin{align}
(-\Delta - k^2)\psi = -q(x)(2e^{ik\cdot x'} + \psi) = 2ik\gamma(x')(2e^{ik\cdot x'} + \psi(x'))\delta(x_3).
\end{align}

Let us denote by $c(x')$ the coefficient of $\delta(x')$ in the right-hand side of (2.3),

\begin{align}
c(x') = 2ik\gamma(x')(2e^{ik\cdot x'} + \psi(x')).
\end{align}

In this notation, (2.3) has the form

\begin{align}
(-\Delta - k^2)\psi = c(x')\delta(x_3).
\end{align}

Observe that coefficient $c(x')$ vanishes outside the support of $\gamma(x')$, and hence the solution of (2.5) can be written as

\begin{align}
\psi(x) = \int_{\text{supp } \gamma} G(x - y')c(y') dy',
\end{align}

where $G(x - y) = \frac{1}{4\pi|x - y|}$ is the Green’s function of (2.5). From (2.6) and (2.4) we obtain equation for determining $c(x')$:

\begin{align}
c(x') + q(x')\int_{\text{supp } \gamma} G(x' - y')c(y') dy' = -2q(x')e^{ik\cdot x'}.
\end{align}
Finally, it is convenient to introduce a modified potential \( v(x') \) as

\[
(2.8) \quad v(x') = \frac{1}{\pi} c(x') e^{-ik \cdot x'}.
\]

Then (2.7) becomes

\[
(2.9) \quad v(x') - 2ik\gamma(x') \int_{\text{supp} \gamma} G(x' - y') e^{ik \cdot (y' - x')} v(y') dy' = \frac{4ik}{\pi} \gamma(x').
\]

Thus, if \( v(x') \) is known, one can find \( \gamma(x) \) from (2.9):

\[
(2.10) \quad \gamma(x) = -\frac{i v(x') k^{-1}}{4\pi^{-1} + 2 \int_{\text{supp} \gamma} G(x' - y') e^{ik \cdot (y' - x')} v(y') dy'}, \quad x' \in \text{supp} \gamma.
\]

In the next section, we will describe a method of determining \( v(x') \) from the far field pattern \( f(k, \hat{x}) \) with fixed \( k \), and this will complete the solution of the inverse impedance problem.

**3. Calculation of the modified potential.** Equation (2.6) contains Green’s function of a shifted argument whose asymptotic behavior has the form

\[
(3.1) \quad G(x - y) = \frac{1}{4\pi} \frac{e^{ik|x|}}{|x|} e^{-ik \cdot \hat{x} \cdot y} \left[ 1 + O \left( \frac{1}{|x|} \right) \right], \quad |x| \to \infty.
\]

Substituting it into (1.4) and (2.6), we obtain the following representation for the far field pattern \( f(k, \hat{x}) \):

\[
(3.2) \quad f(k, \hat{x}) = \frac{1}{4} \int_{\text{supp} \gamma} e^{-i(k \hat{x} \cdot y) \cdot \gamma} v(y') dy',
\]

where \( y' = (y_1, y_2, 0) \) and \( \hat{x} = x/|x| \). Our next goal is to select directions \( \hat{x} \) so that the integral (3.2) would represent the Fourier coefficients of function \( v(y') \).

To this end, we write down the incident vector as \( k = k(\cos \varphi_1, \cos \varphi_2, \cos \varphi_3) \), while \( \hat{x} = (\cos \theta_1, \cos \theta_2, \cos \theta_3) \). Then (3.2) becomes

\[
(3.3) \quad f(k, \hat{x}) = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} e^{-ik[x(\cos \theta_1 - \cos \varphi_1) + y(\cos \theta_2 - \cos \varphi_2)]} v(x, y) dx dy.
\]

Expression (3.3) can be associated with the Fourier coefficients of \( v(x, y) \) if angles \( \theta_1 = \theta_{1,n_1} \) and \( \theta_2 = \theta_{2,n_2} \) are chosen in such a way that

\[
(3.4) \quad k(\cos \theta_{1,n_1} - \cos \varphi_1) = \pi n_1,
\]

\[
(3.5) \quad k(\cos \theta_{2,n_2} - \cos \varphi_2) = \pi n_2,
\]

where \( n_1, n_2 = 0, \pm 1, \pm 2, \ldots \), and

\[
(3.6) \quad \frac{k}{\pi} (1 + \cos \varphi_i) \leq n_i \leq \frac{k}{\pi} (1 - \cos \varphi_i), \quad i = 1, 2.
\]

For those directions defined by the angles \( \theta_{1,n_1} \) and \( \theta_{2,n_2} \), the measured far field pattern \( f_{n_1,n_2} \) will be the Fourier coefficient in the expansion of the modified potential \( v(x, y) \),

\[
(3.7) \quad f_{n_1,n_2} = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} e^{-\pi i(n_1 x + n_2 y)} v(x, y) dx dy.
\]
Hence, \( v(x, y) \) has the following Fourier series representation:

\[
\begin{align*}
v(x, y) &= \sum_{n_1, n_2} f_{n_1n_2} e^{\pi i (n_1x + n_2y)}.
\end{align*}
\] (3.8)

Formula (2.10) along with (3.8) gives the solution of the inverse impedance problem. Observe that only the coefficients \( f_{n_1n_2} \) satisfying (3.6) can be found, and therefore numerical evaluation allows one to find \( v \) only approximately. In fact, condition (3.6) provides \( N \) coefficients, where \((\frac{2k}{\pi})^2 \leq N \leq (\frac{2k}{\pi} + 1)^2\).

4. Asymptotic solution. Now we are going to modify the previous approach assuming \( k \gg 1 \) and using the WKB approximation. The scattered wave \( \psi(x) \) satisfies the Helmholtz equation in the half space \( \mathbb{R}^3_+ = \{ x = (x_1, x_2, x_3), x_3 > 0 \} \),

\[
\begin{align*}
-\Delta \psi &= k^2 \psi, \quad x_3 > 0,
\end{align*}
\] (4.1)

and the boundary condition

\[
\begin{align*}
\psi_{x_3} + ik\gamma(x')\psi(x')\big|_{x_3=0} &= -2ik\gamma(x') e^{ik' x'}, \quad x' = (x_1, x_2, 0).
\end{align*}
\] (4.2)

In order to find the asymptotic behavior of \( \psi(x) \) for large \( k \), we will use the WKB approximation of \( \psi(x) \) in a neighborhood of support of \( \gamma(x') \). We will be looking for an expansion of \( \psi \) in the form

\[
\begin{align*}
\psi(x) &= e^{ik^*\cdot x} \sum_{n=0}^{\infty} \Psi_n(x) (ik)^{-n}, \quad k^* = (k_1, k_2, -k_3).
\end{align*}
\] (4.3)

Coefficients in this expansion can be found explicitly. Substituting (4.3) into (4.1) and equating the coefficients of like powers of \( k \), we obtain a recurrence system of differential equations for \( \Psi_n(x) \):

\[
\begin{align*}
\hat{k}^* \cdot \nabla \Psi_0 &= 0, \\
2\hat{k}^* \cdot \nabla \Psi_n + \Delta \Psi_{n-1} &= 0, \quad n \geq 1,
\end{align*}
\] (4.4) (4.5)

where \( \hat{k}^* \) denotes the unit vector in the direction of vector \( k^* \). From the boundary condition (4.2), one can find the initial condition for \( \Psi_n(x) \) and thus determine all the coefficients \( \Psi_n(x) \). In particular, (4.4) implies

\[
\begin{align*}
\Psi_0(x_1, x_2, x_3) &= \Phi_0 \left( x_1 + \frac{k_1x_3}{k_3}, x_2 + \frac{k_2x_3}{k_3} \right),
\end{align*}
\] (4.6)

where \( \Phi_0 \) is an arbitrary differentiable function. From (4.2) it follows that

\[
\begin{align*}
\Psi_0(x_1, x_2, 0) &= -\frac{2\gamma(x_1, x_2)}{\gamma(x_1, x_2) - k_3 k^{-1}},
\end{align*}
\] (4.7)

and hence

\[
\begin{align*}
\Psi_0(x_1, x_2, x_3) &= -\frac{2\gamma(x_1 + k_1 k_3^{-1} x_3, x_2 + k_2 k_3^{-1} x_3)}{\gamma(x_1 + k_1 k_3^{-1} x_3, x_2 + k_2 k_3^{-1} x_3) - k_3 k^{-1}}.
\end{align*}
\] (4.8)
Thus, using expansion (4.3) and relations (2.4) and (2.8), we obtain the following asymptotic representation for the scattering amplitude \( f \) of (3.2) through the boundary impedance \( \gamma \):

\[
(4.9) \quad f(k, \hat{x}) = -\frac{ik_3}{\pi} \int_{\text{supp } \gamma} e^{-i(k\hat{\theta} - k)\hat{y}} y' \frac{\gamma(y')}{\gamma(y') - k_yk^{-1}} dy' + O(k^{-1}).
\]

If we choose the direction of measurements \( \hat{x} \) of the far field pattern the same as before in (3.4)–(3.6), then the value \( f(k, \hat{x}) \) becomes proportional to the Fourier coefficient of \( \gamma(\gamma - k_3k^{-1})^{-1} \),

\[
(5.10) \quad f_{n_1,n_2} = -\frac{ik_3}{\pi} \int_{-1}^{1} \int_{-1}^{1} e^{-\pi(n_1y_1 + n_2y_2)} \frac{\gamma(y_1, y_2)}{\gamma(y_1, y_2) - k_3k^{-1}} dy_1dy_2 + O(k^{-1}).
\]

Applying the inverse Fourier transform, we obtain

\[
(5.11) \quad \frac{\gamma(x_1, x_2)}{\gamma(x_1, x_2) - k_3k^{-1}} = \frac{\pi i}{4k_3} \sum_{m,n} f_{m,n} e^{\pi i(mx_1 + nx_2)} + O(k^{-1}).
\]

This equation can be solved for \( \gamma \). Hence the boundary impedance can be restored using the values of the far field pattern in the specific directions given by (3.4)–(3.5).

5. Scattering from a sphere. In the case of a convex body \( \Omega \), there is a direct asymptotic relation between the far field pattern and the boundary impedance for large values of \( k \) (see [5]):

\[
(5.1) \quad \gamma(y^+) = \frac{\mathcal{K}^{-\frac{1}{2}}(y^+) + 2f(k, \hat{x})}{\mathcal{K}^{-\frac{1}{2}}(y^+) - 2f(k, \hat{x})} \cdot \hat{x} + O(k^{-1}),
\]

where \( y^+(\hat{x}) \in \partial \Omega \) is the preimage of \( n = (\hat{x} - \hat{k})/|\hat{x} - \hat{k}| \) under the Gauss map and \( \mathcal{K}(y^+) \) is the Gauss curvature at \( y^+ \in \Omega \). Formula (5.1) has asymptotic character, and we want to figure out the range of values of \( k \) that give a good approximation of \( \gamma \). We also analyze the approximation of the far field pattern \( f(k, \hat{x}) \) by the measurement of the scattered field at the distance \( r \).

In order to conduct a numerical experiment, we restrict ourselves to the case where the direct problem can be easily solved. For that purpose we consider the problem that has an exact solution—scattering of plane wave \( e^{ikz} \) from a sphere of radius \( a \) with constant boundary impedance. Then the formula (5.1) takes the form

\[
(5.2) \quad \gamma = \frac{a + 2f(k, \hat{x})}{a - 2f(k, \hat{x})} \sin \frac{\theta}{2} + O(k^{-1}), \quad \frac{\pi}{2} \leq \theta \leq \pi,
\]

where \( \theta \) is the polar angle on the sphere.

Similar to (1.1), we need to solve the boundary value problem for the Helmholtz equation

\[
(5.3) \quad -\Delta u = k^2 u, \quad r > a,
\]

\[
(5.4) \quad u_r - ik\gamma u_r|_{r=a} = 0,
\]

where \( \gamma > 0 \) is a constant surface impedance and

\[
(5.5) \quad u(x) = e^{ikz} + \varphi(x)
\]

with \( \varphi(x) \) satisfying the radiation condition (1.4).
Solution of the problem (5.3)–(5.5) is given by
\begin{equation}
\begin{split}
u &= e^{ikz} - \sum_{n=0}^{\infty} (2n+1) i^n \frac{n j_{n-1}(ka) - (n+1)j_{n+1}(ka) - i\gamma j_n(ka)}{nh_{n-1}(ka) - (n+1)h_{n+1}(ka) - i\gamma h_n(ka)} h_n^{(1)}(kr) P_n(\cos \theta),
\end{split}
\end{equation}
where \(j_n(z)\) and \(h_n^{(1)}(z)\) are spherical Bessel functions of the first and third kind, respectively, and \(P_n(x)\) are Legendre polynomials \cite{8}. From this formula we can determine the far field pattern \(f(k, \hat{x})\) of (1.4) and calculate the surface impedance \(\gamma\) using its asymptotics (5.2) for large values of \(k\).

6. Numerical examples. Figure 6.1 shows the reconstructed boundary impedance \(\gamma = 2\) of a unit sphere \((a = 1)\) based on the asymptotic formula (5.2). In panel (a), the far field pattern with the wave number \(k = 200\) was determined from the exact solution using (1.4),
\begin{equation}
|f(k, \hat{x})| \approx |x|\psi(x),
\end{equation}
for various distances \(|x|\) from the sphere. The accuracy of approximation monotonically improves as the polar angle \(\theta\) increases and does not exceed about 0.5% for the distances beyond \(r = 100a\). Figure 6.1(b) shows the dependence of the restored impedance on the wave number of the incident plane wave while at the distance \(r = 200a\) from the sphere. As the wave number \(k\) decreases, the approximation of \(\gamma\) not only deteriorates, but also starts to exhibit oscillatory behavior. Approximation of \(\gamma\) is also improving for larger values of \(\theta\) and remains below 0.5% as long as \(k > 50\).

The above approach leads to a good approximation of the impedance if the far field is measured at a distance an order of magnitude greater than the diameter of the sphere and if the wave length is an order of magnitude less than the diameter of the sphere.

Fig. 6.1. Reconstruction of the boundary impedance \(\gamma = 2\) of the illuminated part of the sphere of radius \(a = 1\) from (5.2). (a) The far field pattern \(f(k, \hat{x})\) with the wave number \(k = 200\) is approximated by the amplitude of the scattered wave at different distances \(r\) from the center of the sphere. (b) The far field pattern \(f(k, \hat{x})\) is approximated as before with \(r = 200a\); wave numbers \(k\) vary. Deviation from \(\gamma = 2\) increases as angle \(\theta\) approaches 90°, where incident rays are tangent to the sphere.

Similar results are observed in restoring a compactly supported boundary impedance of a half space. The forward problem with the impedance given in Figure 6.2(a) is solved through numerical solution of integral equation (2.10) and application of (3.2). The inverse problem is solved by two different methods. First we reconstruct the boundary impedance using (3.8) and (2.10). Figure 6.2(b) shows the
result when $k = 10$. Then the Fourier coefficients of the far field pattern are perturbed by random numbers uniformly distributed in the interval $[-1,1]$. Figures 6.2(c)–(d) show reconstructed boundary impedance for $k = 15$ when the amplitudes of the additive random noise are 1% and 5%, respectively, of the greatest Fourier coefficient.

The second approach is based on asymptotic formula (4.11), which gives slightly less accuracy as compared with exact formula (2.10). In Figure 6.3 we reconstruct the
boundary impedance from Figure 6.2(a) using asymptotic formula (4.11) and $k = 15$
when the Fourier coefficients of the far field pattern are corrupted by an additive
uniformly distributed random noise from the interval $[-1, 1]$ with the amplitude 1%
(a) and 5% (b) of the largest Fourier coefficient.

Finally, in Figure 6.4 we reconstruct the impedance in the presence of a 1%
additive random noise when the wave number is as small as $k = 5$. Although in this
case there is a significant error in the restored amplitude, it captures qualitatively
the shape and the location of the inhomogeneity of the impedance. All calculations
were performed for specific values of angles $\cos \varphi_1 = \cos \varphi_2 = 1/\sqrt{3}$. Calculations for
different angles give similar results.

7. Conclusions. We have considered the problem of determining a compactly
supported boundary impedance from knowledge of the time-harmonic incident wave
and its far field pattern. The approach is based on a special selection of the directions
in which the far field pattern is measured. Then the boundary impedance is expressed
through a potential using a simple exact formula, while the Fourier coefficients of
the potential equal the measured far field patterns. Efficiency of the approach was
illustrated by numerical examples.

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